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Closure relations for orbits on affine symmetric spaces under the action of parabolic subgroups. Intersections of associated orbits

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§1. Introduction

Let G be a connected Lie group, σ an involution of G and H an open subgroup of $G^\sigma = \{x \in G \mid \sigma x = x\}$. Then the G -homogeneous manifold $H \backslash G$ is called an affine symmetric space. Suppose that G is a real semisimple Lie group. Let P be a minimal parabolic subgroup of G and P' a parabolic subgroup of G containing P . Then the double coset decomposition $H \backslash G / P$ is studied in [2] and [5], the relation between $H \backslash G / P'$ and $H \backslash G / P$ is studied in [3], and the closure relation for $H \backslash G / P$ is studied in [4].

Let θ be a Cartan involution of G such that $\sigma\theta = \theta\sigma$. Put $K = G^\theta$ and let H^a be the open subgroup of $G^{\sigma\theta}$ such that $K \cap H = K \cap H^a$. Then $H^a \backslash G$ is called the affine symmetric space associated to $H \backslash G$. Let A be a θ -stable split component of P and put $U = \{x \in K \mid xAx^{-1} \text{ is } \sigma\text{-stable}\}$.

There exists a natural one-to-one correspondence between the double coset decompositions $H \backslash G / P'$ and $H^a \backslash G / P'$ given by $D \rightarrow D^a = H^a(D \cap U)P'$ for H - P' double cosets D in G ([2], [3]). Moreover it follows easily from Corollary of [4] Theorem that this correspondence reverses the closure relations for the double coset decompositions. In this paper we prove the following theorem.

Theorem. Let D_1 and D_2 be arbitrary H - P' double cosets in G .

$$(i) \quad D_1^{cl} \supset D_2 \iff D_1 \cap D_2^a \neq \emptyset.$$

(ii) Let $I(D_1, D_2)$ be the set of all the H - P' double cosets D in G such that $D_1^{cl} \supset D^{cl} \supset D_2$. Then

$$(D_1 \cap D_2^a)^{cl} \cap D_2^a = \bigcup_{D \in I(D_1, D_2)} D \cap D_2^a.$$

(iii) Let x be an element of U . Then $HxP' \cap H^a xP' = (K \cap H)xP'$.

$$(iv) \quad D_1 \cap D_2^a \text{ is nonempty and closed in } G \iff D_1 = D_2.$$

Example. Let G_1 be a connected semisimple Lie group, θ_1 a Cartan involution of G_1 , $K_1 = \{x \in G_1 \mid \theta_1 x = x\}$, and P_1 a minimal parabolic subgroup of G_1 with a θ_1 -stable split component A_1 . Let P'_1 and P''_1 be parabolic subgroups of G_1 containing P_1 . Put $G = G_1 \times G_1$, $H = \{(x, x) \in G \mid x \in G_1\}$, $H^a = \{(\theta_1 x, x) \in G \mid x \in G_1\}$ and $P' = P'_1 \times P''_1$. Then we have natural bijections

$$H \backslash G / P' \xrightarrow{\sim} P'_1 \backslash G_1 / P''_1$$

and

$$H^a \backslash G / P' \xrightarrow{\sim} \theta_1(P'_1) \backslash G_1 / P''_1$$

by the maps $(x, y) \mapsto x^{-1}y$ and $(x, y) \mapsto \theta_1(x^{-1})y$, respectively. Hence by the Bruhat decomposition of G_1 , every H - P' double coset and H^a - P' double coset have representatives in $W(A_1) \times 1$.

Consider the intersection $I = H(w, 1)P' \cap H^a(w', 1)P'$ for $w, w' \in W(A_1)$. Since $H \cap H^a = \{(x, x) \mid x \in K_1\}$ and since $G_1 = K_1P_1$ by the Iwasawa decomposition of G_1 , I contains elements of the form $(x, 1)$ with $x \in G_1$ if I is nonempty. We have easily

$$(x, 1) \in I \iff x \in P'_1 w P''_1 \cap \theta_1(P'_1) w' P''_1.$$

Thus we have as a corollary of Theorem (i),

$$(1.1) \quad (P'_1 w P''_1)^{cl} \supset P'_1 w' P''_1 \iff P'_1 w P''_1 \cap \theta_1(P'_1) w' P''_1 \neq \emptyset.$$

Especially we have

$$(1.2) \quad (P_1 w P_1)^{cl} \supset P_1 w' P_1 \iff P_1 w P_1 \cap \theta_1(P_1) w' P_1 \neq \emptyset.$$

Remark. In [1], V. V. Deodhar studied explicitly the above type of intersection $P_1 w P_1 \cap \bar{P}_1 w' P_1$ when G_1 is a semisimple algebraic group over an algebraically closed field. Here $\bar{P}_1 = w_0 P_1 w_0^{-1}$ with the longest element w_0 of the Weyl group. He gave (1.2) as a corollary of his results in this case. (Replace $\theta_1(P_1)$ by \bar{P}_1 .)

The author is grateful to J. A. Wolf who suggested him the importance of the intersections of H -orbits and H^a -orbits on G/P . In fact, Theorem (iv) was conjectured by him.

§2. Notations and elementary lemmas

Let \mathfrak{g} be the Lie algebra of G . Let σ and θ be the

involutions of \mathfrak{g} induced from the involutions σ and θ of G , respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{h}^a + \mathfrak{q}^a$ be the decompositions of \mathfrak{g} into the $+1$ and -1 eigenspaces for σ , θ and $\sigma\theta$, respectively.

Let \mathfrak{a} be the Lie algebra of A . Then \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . Let Σ denote the root system of the pair $(\mathfrak{g}, \mathfrak{a})$. Then P can be written as

$$P = P(\mathfrak{a}, \Sigma^+) = Z_G(\mathfrak{a}) \exp \mathfrak{n}$$

with a positive system Σ^+ of Σ . Here $Z_G(\mathfrak{a})$ is the centralizer of \mathfrak{a} in G and $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}(\mathfrak{a}; \alpha)$ ($\mathfrak{g}(\mathfrak{a}; \alpha) = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{a}\}$).

Lemma 1. Let \mathfrak{P}' be a parabolic subalgebra of \mathfrak{g} . Then $\mathfrak{h} + \mathfrak{h}^a + \mathfrak{P}' = \mathfrak{g}$.

Proof. Let X be an element of \mathfrak{P}' . Then

$$\theta X = X - (X + \sigma X) + (\sigma X + \theta X) \subset \mathfrak{P}' + \mathfrak{h} + \mathfrak{h}^a.$$

Hence $\theta \mathfrak{P}' \subset \mathfrak{h} + \mathfrak{h}^a + \mathfrak{P}'$. Since $\mathfrak{P}' + \theta \mathfrak{P}' = \mathfrak{g}$, we have $\mathfrak{h} + \mathfrak{h}^a + \mathfrak{P}' \supset \mathfrak{g}$. Q.E.D.

Lemma 2. Let D_1 and D_2 be arbitrary H - \mathfrak{P}' double cosets in G .

$$(i) \quad (D_1 \cap D_2^a)^{cl} \cap D_2^a = D_1^{cl} \cap D_2^a.$$

$$(ii) \quad D_1^{cl} \supset D_2 \Rightarrow D_1 \cap D_2^a \neq \emptyset.$$

Proof. (i) It is clear that $(D_1 \cap D_2^a)^{cl} \cap D_2^a \subset D_1^{cl} \cap D_2^a$. Let x be an element of $D_1^{cl} \cap D_2^a$. Then we have only to show that $x \in (D_1 \cap D_2^a)^{cl}$. For any neighborhood V of the identity in H^a , the set $HVxP'$ contains a neighborhood of x in G by Lemma 1. Hence $D_1 \cap HVxP' \neq \emptyset$. Since $HD_1P' = D_1$, we have

$$D_1 \cap Vx \neq \emptyset.$$

On the other hand, $Vx \subset D_2^a$. Hence $(D_1 \cap D_2^a) \cap Vx \neq \emptyset$ and we have proved that $x \in (D_1 \cap D_2^a)^{cl}$.

(ii) is clear from (i) since $D_2 \cap D_2^a \neq \emptyset$. Q.E.D.

§3. Proof of Theorem (i) and (ii)

By Lemma 2 (i), Theorem (ii) follows from Theorem (i).

Proof of Theorem (i). By [2] Theorem 1, we can write $D_1 = HxP' \supset HxP$ with $x \in U$. Considering xPx^{-1} and $xP'x^{-1}$ as P and P' , respectively, we may assume that $D_1 = HP'$ and that \underline{a} is σ -stable. By Lemma 2 (ii), we have only to prove the following.

$$D_1 \cap D_2^a \neq \emptyset \Rightarrow D_1^{cl} \supset D_2.$$

Suppose that $D_1 \cap D_2^a \neq \emptyset$. Then $HP \cap D_2^a \neq \emptyset$ since $D_2^a P' = D_2^a$. Hence there exists an element y of $D_2^a \cap U = D_2 \cap U$ such that $HP \cap H^a y P \neq \emptyset$. On the other hand, if $(HP)^{cl} \supset HyP$ for some $y \in D_2$, then it is clear that $D_1^{cl} \supset D_2$. Thus we have only to prove the

following.

(3.1) If $HP \cap H^a_{yP} \neq \emptyset$ for $y \in U$, then $(HP)^{cl} \supset HyP$.

We will prove (3.1) by induction on the real rank of G ($= \dim \mathfrak{a}$). Suppose that $\mathfrak{a} \subset \mathfrak{g}$ and that $\text{Ad}(y)\mathfrak{a} \subset \mathfrak{h}$. Then by [2] Proposition 1 and Proposition 2, HP is open in G and HyP is closed in G . By [4] Proposition, we have always $(HP)^{cl} \supset HyP$. Hence we may assume that

(3.2) $\mathfrak{a} \cap \mathfrak{h} \neq \{0\}$

or that

(3.3) $\text{Ad}(y)\mathfrak{a} \cap \mathfrak{g} \neq \{0\}$.

We first show that the case (3.3) is reduced to the case (3.2). Assume the condition (3.3). Then $\text{Ad}(y)\mathfrak{a} \cap \mathfrak{h}^a \neq \{0\}$ since $\text{Ad}(y)\mathfrak{a} \subset \mathfrak{p}$. Consider $\text{Ad}(y)\mathfrak{a}$, \mathfrak{h}^a and yPy^{-1} as \mathfrak{a} , \mathfrak{h} and P in the case (3.2), respectively. Then we have in the proof of the case (3.2) that

$$H^a_{yPy^{-1}} \cap H^a_{Py^{-1}} \neq \emptyset \Rightarrow (H^a_{yPy^{-1}})^{cl} \supset H^a_{Py^{-1}}.$$

Hence

$$HP \cap H^a_{yP} \neq \emptyset \Rightarrow (H^a_{yP})^{cl} \supset H^a_P.$$

On the other hand, we have

$$(HP)^{cl} \supset HyP \iff (H^a_{yP})^{cl} \supset H^a_P$$

for $y \in U$ by Corollary of [4] Theorem. Thus the case (3.3) is reduced to the case (3.2) and so we may assume (3.2) in the

followings.

By [4] Theorem (iv), there exists a sequence $\alpha_1, \dots, \alpha_n$ of simple roots in Σ^+ such that

$$(3.4) \quad (HP)^{cl} = H((L \cap H)(L \cap wPw^{-1}))^{cl} wP L_{\alpha_n} \dots L_{\alpha_1}.$$

Here $w = w_{\alpha_1} \dots w_{\alpha_n}$, L is the analytic subgroup of G for $\underline{l} = [\underline{z}_{\underline{g}}(\underline{a} \cap \underline{h}), \underline{z}_{\underline{g}}(\underline{a} \cap \underline{h})]$ and $L_{\alpha} = Z_G(\underline{a}^{\alpha})$, $\underline{a}^{\alpha} = \{Y \in \underline{a} \mid \alpha(Y) = 0\}$ for $\alpha \in \Sigma$.

Lemma 3. $HwP = (K \cap H)(L \cap H)_0 wP$. ($(L \cap H)_0$ is the connected component of $L \cap H$ containing the identity.)

Proof. Put $L_1 = Z_G(\underline{a} \cap \underline{h})$ and define a parabolic subgroup P_1 of G by $P_1 = L_1 wPw^{-1}$ as in [4] §1. Then $P_1 \cap H_0$ is a parabolic subgroup of H_0 and we have $H_0 = (K \cap H)_0 (P_1 \cap H)_0$ by the Iwasawa decomposition of H_0 . On the other hand, $K \cap H$ intersects with every connected component of H since $H = (K \cap H) \exp(\underline{p} \cap \underline{h})$. Hence

$$(3.5) \quad H = (K \cap H)(P_1 \cap H)_0.$$

Let \underline{n}_1 be the nilpotent radical of the Lie algebra of P_1 . Then $P_1 = L_1 \exp \underline{n}_1$ is a Langlands decomposition of P_1 . Since L_1 and \underline{n}_1 are σ -stable, we have

$$(3.6) \quad (P_1 \cap H)_0 = (L_1 \cap H)_0 \exp(\underline{n}_1 \cap \underline{h}).$$

Let \underline{z} be the center of the Lie algebra \underline{l}_1 of L_1 . Then $\underline{l}_1 = \underline{z} + \underline{l}$. Since \underline{z} and \underline{l} are σ -stable, we have $\underline{l}_1 \cap \underline{h} = \underline{z} \cap \underline{h} +$

$\underline{1} \cap \underline{h}$ and therefore

$$(3.7) \quad (L_1 \cap H)_0 = (L \cap H)_0 \exp(\underline{z} \cap \underline{h}).$$

We get the desired formula from (3.5), (3.6) and (3.7) since $\exp \underline{n}_1 \subset wPw^{-1}$ and $\exp \underline{z} \subset wPw^{-1}$. Q.E.D.

Now we will continue the proof of Theorem (i). Suppose that $HP \cap H^a yP \neq \emptyset$. Since $HP \subset HwPL_{\alpha_n} \dots L_{\alpha_1}$, we have

$$HwP \cap H^a yPL_{\alpha_1} \dots L_{\alpha_n} \neq \emptyset$$

By Lemma 3, we have

$$(3.8) \quad (L \cap H)_0 \cap H^a yPL_{\alpha_1} \dots L_{\alpha_n} w^{-1} \neq \emptyset.$$

Let y' be an element of the left hand side of (3.8) and y'' an element of $(L \cap H^a)_0 y' (L \cap wPw^{-1}) \cap U$. Then

$$(3.9) \quad H^a y'' wP \subset H^a yPL_{\alpha_1} \dots L_{\alpha_n}$$

and

$$(3.10) \quad (L \cap H)_0 (L \cap wPw^{-1}) \cap (L \cap H^a)_0 y'' (L \cap wPw^{-1}) \neq \emptyset.$$

Since $\sigma L = \theta L = L$ and $\dim(\underline{1} \cap \underline{a}) < \dim \underline{a}$, we have

$$((L \cap H)_0 (L \cap wPw^{-1}))^{cl} \supset (L \cap H)_0 y'' (L \cap wPw^{-1})$$

by the assumption of induction. By (3.4), we have

$$(3.11) \quad \begin{aligned} (HP)^{cl} &\supset H(L \cap H)_0 y'' (L \cap wPw^{-1}) wPL_{\alpha_n} \dots L_{\alpha_1} \\ &\supset Hy'' wPL_{\alpha_n} \dots L_{\alpha_1}. \end{aligned}$$

Now consider the formula (3.9) which can be rewritten as

$$y \in H^a y''^{wPL_{\alpha_n} \dots L_{\alpha_1}}.$$

As in the proof of [4] Theorem (vi), we can choose a $y_1 \in y''^{wPL_{\alpha_n} \dots L_{\alpha_1}} \cap U$ so that $y \in H^a y_1 P$. Since $y \in U$, it follows from [2] Theorem 1 that $y \in (K \cap H) y_1 P$. Hence

$$(3.12) \quad y \in (K \cap H) y''^{wPL_{\alpha_n} \dots L_{\alpha_1}}.$$

From (3.11) and (3.12), we have $(HP)^{cl} \supset HyP$ as desired.

Q.E.D.

§4. Proof of Theorem (iii) and (iv)

Theorem (iv) follows from (ii) and (iii). So we have only to prove (iii) in this section. Recall the definition of $P = P(\underline{a}, \Sigma^+)$ in §2 and let Ψ denote the set of all the simple roots in Σ^+ .

Lemma 4. Suppose that $H^a P$ is not open in G . Then there exists an $\alpha \in \Psi$ such that $\dim H^a P_\alpha > \dim H^a P$.

Proof. By [2] Theorem 1, we may assume that $\sigma \underline{a} = \underline{a}$. By [2] Proposition 1, Σ^+ is not σ -compatible or $\underline{a} \cap \underline{h}$ is not maximal abelian in $\underline{p} \cap \underline{h}$. First suppose that Σ^+ is not σ -compatible. Then by [4] Lemma 4 and Lemma 5, there exists an $\alpha \in \Psi$ such that $H^a P_\alpha = H^a P \cup H^a w_\alpha P$ and that $\dim H^a w_\alpha P > \dim H^a P$. Hence we may assume that Σ^+ is σ -compatible and that $\underline{a} \cap \underline{h}$ is not maximal abelian in $\underline{p} \cap \underline{h}$.

Put $\underline{l}_1 = \underline{z}_{\underline{q}}(\underline{a} \cap \underline{h})$. Suppose that there exists an $\alpha \in \Psi \cap \Sigma(\underline{l}_1; \underline{a})$ such that $\underline{q}(\underline{a}; \alpha) \cap \underline{q}^a \neq \{0\}$. Here $\Sigma(\underline{l}_1; \underline{a})$ is the root system of the pair $(\underline{l}_1, \underline{a})$, and it is clear that $\alpha \in \Sigma(\underline{l}_1; \underline{a})$ if and only if $\alpha \in \Sigma$, $\sigma\alpha = -\alpha$. Then by [4] Lemma 3 (F), $\dim H^a P_\alpha > \dim H^a P$. Hence we may assume that

$$(4.1) \quad \underline{q}(\underline{a}; \alpha) \cap \underline{q}^a = \{0\} \quad \text{for all } \alpha \in \Psi \cap \Sigma(\underline{l}_1; \underline{a}).$$

Let β be a root in $\Sigma(\underline{l}_1; \underline{a}) \cap \Sigma^+$ and write $\beta = \sum_{\alpha \in \Psi} n_\alpha \alpha$. Choose an element $Y \in \underline{a} \cap \underline{h}$ such that $\alpha(Y) > 0$ for all $\alpha \in \Sigma^+ - \Sigma(\underline{l}_1; \underline{a})$ by [4] Lemma 4. If $n_\alpha > 0$ for some $\alpha \in \Psi - \Sigma(\underline{l}_1; \underline{a})$, then $\beta(Y) > 0$. But since $\beta(Y) = 0$, we have proved that β is written as a linear combination of roots in $\Psi \cap \Sigma(\underline{l}_1; \underline{a})$. By (4.1) and [4] Lemma 10, we have $\underline{q}(\underline{a}; \beta) \subset \underline{h}^a$. Hence

$$\underline{a} \cap \underline{q} + \sum_{\beta \in \Sigma(\underline{l}_1; \underline{a})} \underline{q}(\underline{a}; \beta) \subset \underline{h}^a.$$

Since $\underline{z}_{\underline{q}}(\underline{a} \cap \underline{h}) = \underline{l}_1 = \underline{a} + \sum_{\beta \in \Sigma(\underline{l}_1; \underline{a})} \underline{q}(\underline{a}; \beta)$, $\underline{a} \cap \underline{h}$ is a maximal abelian subspace of $\underline{p} \cap \underline{h} = \underline{p} \cap \underline{q}^a$. But this is a contradiction to the assumption on $\underline{a} \cap \underline{h}$. Q.E.D.

Lemma 5. If HP is closed in G , then $HP = (K \cap H)P$.

Proof. If $HP = (K \cap H)xP$ for some $x \in HP$, then $HP = (K \cap H)P$. So taking a conjugate of P , we may assume that $\sigma \underline{a} = \underline{a}$. Since Σ^+ is σ -compatible, we can apply Lemma 3 for $w = 1$ to get

$$HP = (K \cap H)(L \cap H)_0 P.$$

Since $\underline{a} \cap \underline{h}$ is maximal abelian in $\underline{p} \cap \underline{h}$, we have $L \subset H^a$ by the

argument in [4] Proof of Lemma 6 (i). Since $H \cap H^a = K \cap H$, we have $HP = (K \cap H)(L \cap H^a \cap H)_0 P = (K \cap H)P$ as desired. Q.E.D.

Proof of Theorem (iii). Choose $x' \in xP' \cap U$ so that $Hx'P$ has the minimum dimension among the H - P double cosets contained in HxP' . Clearly $HxP' \cap H^a xP' = (HxP' \cap H^a x'P)P'$. Since $(Hx'P)^{cl} \cap HxP' = Hx'P$, it follows from Theorem (i) that $HxP' \cap H^a x'P = Hx'P \cap H^a x'P$. So we have only to prove that

$$(4.2) \quad Hx'P \cap H^a x'P = (K \cap H)x'P \quad \text{for } x' \in U.$$

We will prove (4.2) by induction on the codimension of $H^a x'P$. Rewriting $x'Px'^{-1}$ by P , we may assume that $x' = 1$ and that $\sigma_{\underline{a}} = \underline{a}$.

Suppose that $H^a P$ is open in G . Then HP is closed in G by [2] §3 Corollary and $HP = (K \cap H)P \subset H^a P$ by Lemma 5. Hence we may assume that $H^a P$ is not open in G .

By Lemma 4, there exists an $\alpha \in \Psi$ such that $\dim H^a P_\alpha > \dim H^a P$. Then by [4] Lemma 3, there are two cases $(B^a) : \sigma\theta\alpha \neq \pm\alpha$, $\sigma\theta\alpha \in \Sigma^+$ and $(D^a) : \sigma\theta\alpha = \alpha$, $\underline{q}(\underline{a}; \alpha) \cap \underline{q}^a \neq \{0\}$. Put $z = w_\alpha$ in the case (B^a) and put $z = c_\alpha$ in the case (D^a) . Then we have $(HzP)^{cl} \cap HP_\alpha = HzP$ by [4] Lemma 3 (A) and (F). (Since $\theta|_{\underline{a}} = -1$, we have $(B^a) = (A) : \sigma\alpha \neq \pm\alpha$, $\sigma\alpha \notin \Sigma^+$ and $(D^a) = (F) : \sigma\alpha = -\alpha$, $\underline{q}(\underline{a}; \alpha) \cap \underline{q}^a \neq \{0\}$.) Applying Theorem (i), we have

$$(4.3) \quad HzP \cap H^a P_\alpha = HzP \cap H^a zP.$$

Let y be an element of $HP \cap H^a P$. Then we have only to show

that $y \in (K \cap H)P$ since it is clear that $(K \cap H)P \subset HP \cap H^aP$. Let y' be an element of $H^aP \cap yP_\alpha$. Then by (4.3) and the assumption of induction, we have

$$y' \in H^aP \cap H^aP_\alpha \cap yP_\alpha = H^aP \cap H^aP_\alpha \cap yP_\alpha = (K \cap H)P_\alpha \cap yP_\alpha$$

and therefore

$$y \in (K \cap H)P_\alpha = (K \cap H)P_\alpha.$$

Since $y \in H^aP$, we have

$$y \in (K \cap H)P_\alpha \cap H^aP = (K \cap H)(P_\alpha \cap H^a)P = (K \cap H)JP.$$

Here J is the image of $P_\alpha \cap H^a$ under the projection $P_\alpha \rightarrow L_\alpha$ with respect of the Langlands decomposition $P_\alpha = L_\alpha \exp \underline{n}_\alpha$. We consider the two cases (B^a) and (D^a) separately.

First consider the case (B^a) . We have only to show that $J \subset L_\alpha \cap P$. Let L_α^S denote the identity component of the semisimple part of L_α as in [4] §3. Since $L_\alpha^S \cap J \supset \exp(\underline{q}(\underline{a}; \alpha) + \underline{q}(\underline{a}; 2\alpha))$, we have $L_\alpha^S - (L_\alpha^S \cap J)w_\alpha(L_\alpha^S \cap P) \subset L_\alpha^S \cap P$ by the Bruhat decomposition of L_α^S . Since $L_\alpha/L_\alpha \cap P \cong L_\alpha^S/L_\alpha^S \cap P$, we have $L_\alpha - Jw_\alpha(L_\alpha \cap P) \subset L_\alpha \cap P$. On the other hand, we have $J(L_\alpha \cap P) \cap Jw_\alpha(L_\alpha \cap P) = \emptyset$ since $H^aP \cap H^aP_\alpha \neq \emptyset$. Hence $J \subset L_\alpha \cap P$.

Next consider the case (D^a) . We have only to show that $J \subset (K \cap H)(L_\alpha \cap P)$. In this case, $J \supset L_\alpha^S \cap H^a$ and it follows easily from the proof of [4] Lemma 3 (D) that

$$L_\alpha = D(1) \cup D(w_\alpha) \cup D(c_\alpha) \cup D(c_\alpha^{-1}).$$

Here $D(x) = (L_\alpha^S \cap H^a)x(L_\alpha \cap P)$ for $x \in L_\alpha$. We also have

$$(4.4) \quad J(L_\alpha \cap P) = \begin{cases} D(1) & \text{if } w_\alpha \notin N_{K \cap H}(\underline{a})Z_K(\underline{a}) \\ D(1) \cup D(w_\alpha) & \text{if } w_\alpha \in N_{K \cap H}(\underline{a})Z_K(\underline{a}) \end{cases}$$

since $(H^a P \cup H^a w_\alpha P) \cap (H^a c_\alpha P \cup H^a c_\alpha^{-1} P) = \emptyset$. Since $D(1)$ and $D(w_\alpha)$ are closed in L_α , we have

$$(4.5) \quad D(x) = (L_\alpha^S \cap K \cap H)x(L_\alpha \cap P) \quad \text{for } x = 1 \text{ and } w_\alpha$$

by Lemma 5. (Note that $L_\alpha/L_\alpha \cap P \simeq L_\alpha^S/L_\alpha^S \cap P$.) From (4.4) and (4.5), we get

$$J(L_\alpha \cap P) \subset (K \cap H)(L_\alpha \cap P)$$

as desired.

Q.E.D.

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